Evaluate $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$. Does $\lim_{x\to 2} f(x)$ exist ? If so, what is it ? If not, why not ?

3. Let $f(x) = \begin{cases} x^3, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$

Evaluate $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$. Does $\lim_{x\to 1} f(x)$ exist ? If so, what is it ? If not, why not ?

4. Let $f(x) = \begin{cases} 1-x^2, & \text{if } x \neq 1 \\ 2, & \text{if } x = 1 \end{cases}$

Evaluate $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$. Does $\lim_{x\to 1} f(x)$ exist ? If so, what is it ? If not, why not ?

ANSWERS

1. $\lim_{x\to 2^+} f(x) = 2$, $\lim_{x\to 2^-} f(x) = 1$; No, because $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$

2. $\lim_{x\to 2^+} f(x) = 1$, $\lim_{x\to 2^-} f(x) = 1$; yes, $\lim_{x\to 2} f(x) = 1$

3. $\lim_{x\to 1^-} f(x) = 1$, $\lim_{x\to 1^-} f(x) = 1$; yes, $\lim_{x\to 1} f(x) = 1$

4. $\lim_{x\to 1^+} f(x) = 0$, $\lim_{x\to 1^-} f(x) = 0$; yes, $\lim_{x\to 1} f(x) = 0$

1.5 LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

So far we have considered limits as x approaches a finite number a. In this section we consider limits where x approaches ∞ or $-\infty$. Such limits are referred to as **limits at infinity**. These limits determine what is called the *end behaviour* of a function. For example, consider the behaviour of the function $f(x) = \frac{1}{x}$ as x gets "larger and larger". If we investigate the graph of f (see Figure 1.6), we see that as x increases without bound through positive values, the values of $f(x)$ approach 0. This statement is expressed symbolically as $\lim_{x\to\infty} \frac{1}{x} = 0$. Similarly, as x decreases without bound through negative values, the values of $f(x)$ approach 0. This statement is expressed symbolically as $\lim_{x\to -\infty} \frac{1}{x} = 0$.

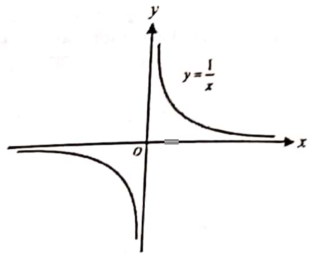

FIGURE 1.6

In general, we use the notation

$$\lim_{x\to\infty} f(x) = L$$

to indicate that as x increases without bound through positive values, the values of $f(x)$ get arbitrarily close to the number L. In this case, the line $y = L$ is called a **horizontal asymptote** of the graph of f (Figure 1.7).

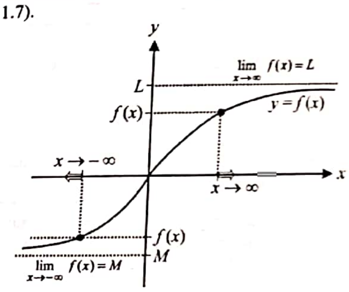

FIGURE 1.7

Similarly, we use the notation

$$\lim_{x\to -\infty} f(x) = M$$

to indicate that as x decreases without bound through negative values, the values of $f(x)$ get arbitrarily close to the number M. In this case, the line $y = M$ is the horizontal asymptote of the graph of f (Figure 1.7).

DEFINITION Horizontal asymptote
A line $y = b$ is called a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to -\infty} f(x) = b$$

Note Note that from the above definition, it follows that the graph of a function can have at most two horizontal asymptotes – one to the right and one to the left.

EXAMPLE 19 The curve $y = f(x)$ shown in Figure 1.8 has two horizontal asymptotes, namely $y = 2$ and $y = -3$ because

$$\lim_{x \to -\infty} f(x) = 2 \quad \text{and} \quad \lim_{x \to \infty} f(x) = -3$$

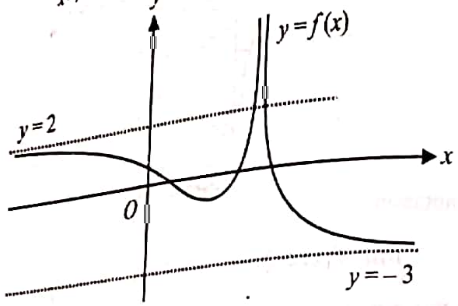

FIGURE 1.8

Precise Definitions of Limit as x Approaches ∞ or −∞

DEFINITION **Limit as x approaches ∞**
Let f be a function defined on an interval (a, ∞). We say that $f(x)$ has the **limit L as x approaches ∞**, and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a corresponding number $M > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > M$$

The definition of a limit as $x \to \infty$ is shown in Figure 1.9. In this figure, note that for a given positive number ε, there exists a positive number M such that, for $x > M$, the graph of f lies between the horizontal lines given by $y = L - \varepsilon$ and $y = L + \varepsilon$.

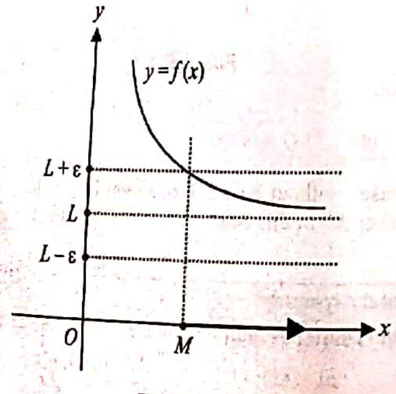

FIGURE 1.9

DEFINITION **Limit as x approaches $-\infty$**
Let f be a function defined on an interval $(-\infty, a)$. We say that $f(x)$ has the **limit L as x approaches $-\infty$**, and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a corresponding number $N < 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x < N$$

The definition of a limit as $x \to -\infty$ is shown in Figure 1.10. In this figure, note that for a given positive number ε, there exists a negative number N such that, for $x < N$, the graph of f lies between the horizontal lines given by $y = L - \varepsilon$ and $y = L + \varepsilon$.

Intuitively, the statement $\lim_{x \to \infty} f(x) = L$ means that as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L. Similarly, the statement $\lim_{x \to -\infty} f(x) = L$ means that as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L.

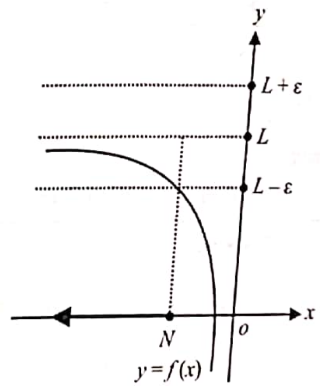

FIGURE 1.10

EXAMPLE 20 Prove that

(a) $\lim_{x \to \infty} \frac{1}{x} = 0$ (b) $\lim_{x \to -\infty} \frac{1}{x} = 0$

SOLUTION (a) Let $\varepsilon > 0$ be given. We must find a number $M > 0$ such that for all x,

$$\text{if} \quad x > M \quad \text{then} \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon$$

We may assume that $x > 0$. Then

$$\left|\frac{1}{x}\right| = \frac{1}{x} < \varepsilon \quad \text{iff} \quad x > \frac{1}{\varepsilon}$$

Thus, if we choose $M = \frac{1}{\varepsilon}$, then for all x

$$x > M \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \frac{1}{x} < \varepsilon$$

It follows that $\lim_{x\to\infty} \frac{1}{x} = 0$

(*b*) Let $\varepsilon > 0$ be given. We must find a number $N < 0$ such that for all x,

if $\quad x < N \quad$ then $\quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon$

We may assume that $x < 0$. Then

$$\left|\frac{1}{x}\right| = -\frac{1}{x} < \varepsilon \quad \text{iff} \quad x < -\frac{1}{\varepsilon}$$

Thus, if we choose $N = -\frac{1}{\varepsilon}$, then for all x

$$x < N \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| = -\frac{1}{x} < \varepsilon$$

It follows that $\lim_{x\to-\infty} \frac{1}{x} = 0$

Properties of Limits at Infinity

Most of the properties of limits that were given in Section 1.2 also hold for limits at infinity.

THEOREM 1.4 Limit Laws as $x \to \pm\infty$

If L, M, and k, are real numbers and $\lim_{x\to\pm\infty} f(x) = L$ and $\lim_{x\to\pm\infty} g(x) = M$, then

1. *Sum Rule* : $\lim_{x\to\pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule* : $\lim_{x\to\pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule* : $\lim_{x\to\pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule* : $\lim_{x\to\pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule* : $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
6. *Power Rule* : If r and s are integers with no common factors, $s \neq 0$, then $\lim_{x\to\pm\infty} (f(x))^{r/s} = L^{r/s}$, provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

EXAMPLE 21 Evaluate each of the following limits :

(*i*) $\lim_{x\to\infty}\left(7 + \frac{5}{x}\right)$ (*ii*) $\lim_{x\to\infty}\left(5 - \frac{1}{x^2}\right)$ (*iii*) $\lim_{x\to-\infty} \frac{\pi\sqrt{2}}{x^3}$

SOLUTION (*i*) $\lim_{x\to\infty}\left(7 + \frac{5}{x}\right) = \lim_{x\to\infty} 7 + \lim_{x\to\infty} \frac{5}{x} = 7 + 5\lim_{x\to\infty} \frac{1}{x} = 7 + 5(0) = 7$

(*ii*) $\lim_{x\to\infty}\left(5 - \frac{1}{x^2}\right) = \lim_{x\to\infty} 5 - \lim_{x\to\infty} \frac{1}{x^2} = 5 - \lim_{x\to\infty}\left(\frac{1}{x}\cdot\frac{1}{x}\right) = 5 - \lim_{x\to\infty}\frac{1}{x}\cdot\lim_{x\to\infty}\frac{1}{x} = 5 - 0\cdot 0 = 5$

(*iii*) $\lim_{x\to-\infty} \frac{\pi\sqrt{2}}{x^3} = \lim_{x\to-\infty}\left(\pi\sqrt{2}\cdot\frac{1}{x}\cdot\frac{1}{x}\cdot\frac{1}{x}\right) = \pi\sqrt{2}\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x} = \pi\sqrt{2}\cdot 0\cdot 0\cdot 0 = 0$

Limits at Infinity of Rational Functions

To evaluate the limit of a rational function, we divide both the numerator and denominator by the highest power of x that appears in the denominator. What happens then depends on the degrees of the polynomials involved. When evaluating limits at infinity, the following result is very useful.

A useful result. If n is a positive integer and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^n} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{c}{x^n} = 0$$

EXAMPLE 22 Evaluate each of the following limits.

(*i*) $\lim_{x\to\infty} \frac{2x^2 + 5x + 1}{3x^3 - 2x + 4}$ (*ii*) $\lim_{x\to\infty} \frac{2x^2 - 5x - 3}{3x^2 - x - 20}$ (*iii*) $\lim_{x\to\infty} \frac{2x^2 + 5x - 1}{x + 2}$

SOLUTION (*i*) As x approaches ∞, both the numerator and denominator approach ∞ and therefore the given function takes the indeterminate form $\frac{\infty}{\infty}$. However, we can change the form of the quotient so that a conclusion can be drawn as to whether or not it has a limit. This is done by dividing both numerator and denominator by the highest power of x that occurs in the denominator. Thus dividing both the numerator and denominator by x^3, we get

$$\lim_{x\to\infty} \frac{2x^2 + 5x + 1}{3x^3 - 2x + 4} = \lim_{x\to\infty} \frac{\frac{2}{x} + \frac{5}{x^2} + \frac{1}{x^3}}{3 - \frac{2}{x^2} + \frac{4}{x^3}}$$

$$= \frac{\lim_{x\to\infty}\frac{2}{x} + \lim_{x\to\infty}\frac{5}{x^2} + \lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{2}{x^2} + \lim_{x\to\infty}\frac{4}{x^3}} = \frac{0+0+0}{3-0+0} = \frac{0}{3} = 0.$$

A similar calculation shows that $\lim_{x\to-\infty} \dfrac{2x^2+5x+1}{3x^3-2x+4} = 0.$

Note that the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator.

(*ii*) Again we divide both the numerator and denominator by the highest power of x appearing in the denominator, which is x^2:

$$\lim_{x\to\infty} \frac{2x^2-5x-3}{3x^2-x-20} = \lim_{x\to\infty} \frac{2-\frac{5}{x}-\frac{3}{x^2}}{3-\frac{1}{x}-\frac{20}{x^2}} = \frac{2-0-0}{3-0-0} = \frac{2}{3}.$$

Similarly, it can be shown that $\lim_{x\to-\infty} \dfrac{2x^2-5x-3}{3x^2-x-20} = \dfrac{2}{3}.$

In this case, the degree of the polynomial in the numerator equals the degree of the polynomial in the denominator.

(*iii*) Dividing both the numerator and denominator by the highest power of x appearing in the denominator, we get

$$\lim_{x\to\infty} \frac{2x^2+5x-1}{x+2} = \lim_{x\to\infty} \frac{2x+5-\frac{1}{x}}{1+\frac{2}{x}} = \infty.$$

A similar analysis shows that $\lim_{x\to-\infty} \dfrac{2x^2+5x-1}{x+2} = -\infty$

In this case, the degree of the polynomial in the numerator is greater than the degree of the polynomial in the denominator.

The conclusions obtained in the preceding example can be generalized to all rational functions as stated in the following theorem.

THEOREM 1.5 Limits at Infinity and Horizontal Asymptotes of Rational Functions

Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function, where

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \ldots\ldots + a_0 \quad \text{and} \quad q(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots\ldots + b_0$$

with $a_m \neq 0$ and $b_n \neq 0$.

- If $m < n$, then $\lim_{x\to\pm\infty} f(x) = 0$. That is, if the degree of numerator is less than the degree of denominator, then the limit of the rational function is 0. In this case, the line $y = 0$ is a horizontal asymptote of f.

- If $m = n$, then $\lim_{x\to\pm\infty} f(x) = \dfrac{a_m}{b_n}$. That is, if the degree of numerator is equal to the degree of denominator, then the limit of the rational function is the ratio of the leading coefficients. In this case, the line $y = a_m/b_n$ is a horizontal asymptote of f.
- If $m > n$, then $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$. That is, if the degree of numerator is greater than the degree of denominator, then the limit of the rational function does not exist. In this case, f has no horizontal asymptote.

EXAMPLE 23 Evaluate $\lim_{x\to\infty} \dfrac{5x^2+8x-3}{3x^2+2}$ and identify any horizontal asymptotes.

SOLUTION

$$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}} = \frac{\lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{8}{x} - \lim_{x\to\infty} \frac{3}{x^2}}{\lim_{x\to\infty} 3 + \lim_{x\to\infty} \frac{2}{x^2}}$$

$$= \frac{5+0-0}{3+0} = \frac{5}{3}$$

Thus the graph of $f(x) = \dfrac{5x^2+8x-3}{3x^2+2}$ has the line $y = \dfrac{5}{3}$ as a horizontal asymptote on the right.

Similarly, it can be shown that $\lim_{x\to-\infty} \dfrac{5x^2+8x-3}{3x^2+2} = \dfrac{5}{3}$. Thus the graph of $f(x) = \dfrac{5x^2+8x-3}{3x^2+2}$ has also the line $y = 5/3$ as a horizontal asymptote on the left.

The graph of $f(x) = \dfrac{5x^2+8x-3}{3x^2+2}$ is sketched in Figure 1.11.

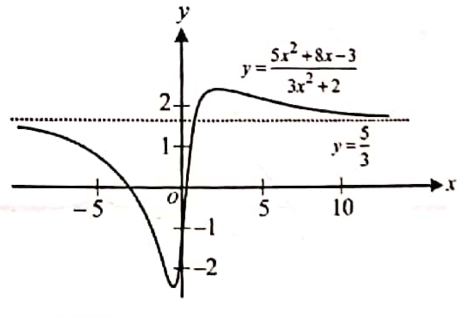

FIGURE 1.11

Note Whenever a horizontal asymptote of a rational function $f(x) = \dfrac{p(x)}{q(x)}$ exists, we always have

$$\lim_{x\to\infty}\frac{p(x)}{q(x)} = \lim_{x\to-\infty}\frac{p(x)}{q(x)}.$$

However, it is not true, in general, for other functions, as shown in the next example.

EXAMPLE 24 Let $f(x) = \frac{x}{\sqrt{x^2+1}}$. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ and then identify any horizontal asymptotes.

SOLUTION First, we evaluate the limit as $x\to\infty$. Dividing the numerator and denominator by $\sqrt{x^2}=x$ for $x\ge 0$, we get

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1$$

Thus, the graph of f has the line $y = 1$ as a horizontal asymptote on the right.

Next, we evaluate the limit as $x\to-\infty$. Dividing the numerator and denominator by $\sqrt{x^2}=-x$ for $x<0$, we get

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty}\frac{\frac{x}{-x}}{\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to-\infty}\frac{-1}{\sqrt{1+\frac{1}{x^2}}} = -1$$

Thus, the graph of f has the line $y = -1$ as a horizontal asymptote on the left.

EXAMPLE 25 Find the horizontal asymptote of the function $f(x) = 2 + \frac{\sin x}{x}$.

SOLUTION We need to investigate the behaviour of the function as $x\to\pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$$

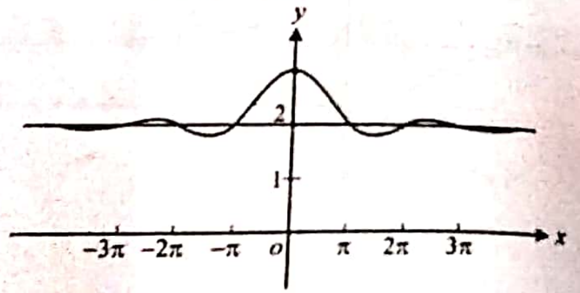

FIGURE 1.12

and since $\lim_{x\to\pm\infty}\left|\frac{1}{x}\right| = 0$, it follows, by the Squeeze Theorem, that $\lim_{x\to\pm\infty}\frac{\sin x}{x} = 0$.

Hence
$$\lim_{x\to\pm\infty}\left(2+\frac{\sin x}{x}\right) = \lim_{x\to\pm\infty} 2 + \lim_{x\to\pm\infty}\frac{\sin x}{x} = 2 + 0 = 2$$

Thus, the line $y = 2$ is a horizontal asymptote of the curve on both right and left (see Figure 1.12).

Note The graph of the above function intersects its horizontal asymptotes infinitely many times. Sometimes it is easier to find the limit at infinity by making a suitable substitution. This is illustrated in the following example.

EXAMPLE 26 Evaluate $\lim_{x\to\infty}\sin\frac{1}{x}$.

SOLUTION If we let $t = \frac{1}{x}$, then $t\to 0^+$ as $x\to\infty$. Therefore,

$$\lim_{x\to\infty}\sin\frac{1}{x} = \lim_{t\to 0^+}\sin t = 0$$

EXAMPLE 27 Evaluate $\lim_{x\to\infty}(\sqrt{x^2+1}-x)$.

SOLUTION
$$\lim_{x\to\infty}(\sqrt{x^2+1}-x) = \lim_{x\to\infty}\left[(\sqrt{x^2+1}-x)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right]$$

$$= \lim_{x\to\infty}\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}+x}$$

$$= \lim_{x\to\infty}\frac{1}{x\left(\sqrt{1+\frac{1}{x^2}}\right)+x}$$

$$= \lim_{x\to\infty}\frac{1}{x\left(\sqrt{1+\frac{1}{x^2}}+1\right)}$$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \lim_{x \to \infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}+1} = 0 \times \frac{1}{2} = 0.$$

1.6 INFINITE LIMITS AND VERTICAL ASYMPTOTES

In this section we extend the concept of limit to infinite limits. An infinite limit occurs when the values of the function increase or decrease without bound near a point. For example, consider the function f defined by $f(x) = \frac{1}{x}$. Let us discuss $\lim_{x \to 0} f(x)$ *i.e.* the limit of $f(x)$ as x approaches 0. To examine the limit of $f(x)$ as x approaches 0, let us find values of $f(x)$ for some values of x that are very close to but unequal to 0. These are given in Table 1.2.

TABLE 1.2

$x > 0$			$x < 0$		
$f(0.1)$	=	10	$f(-0.1)$	=	-10
$f(0.01)$	=	100	$f(-0.01)$	=	-100
$f(0.001)$	=	1000	$f(-0.001)$	=	-1000
$f(0.0001)$	=	10000	$f(-0.0001)$	=	-10000
$f(0.00001)$	=	100000	$f(-0.00001)$	=	-100000

It is evident from the table and graph in Figure 1.13 that as x approaches 0 from the right, the values of $f(x) = \frac{1}{x}$ are positive and increase without bound. We describe this limit behaviour by writing

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

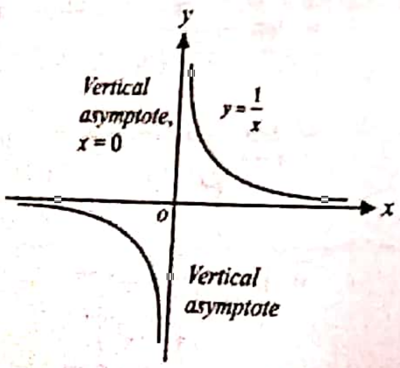

FIGURE 1.13

The infinite symbol indicates that $f(x)$ grows arbitrarily large and positive as x approaches 0^+.

Similarly, as x approaches 0 from the left, the values of $f(x) = \frac{1}{x}$ are negative and decrease without bound. We describe this limit behaviour by writing

$$\lim_{x \to 0^-} f(x) = -\infty$$

The negative infinite symbol indicates that $f(x)$ becomes arbitrarily large and negative as $x \to 0^-$.

Informal Definitions of Infinite Limits

DEFINITION Infinite Limits
Suppose f is defined for all x near a. We say that **the limit of $f(x)$ as x approaches a is infinite**, and write

$$\lim_{x \to a} f(x) = \infty$$

if the values of $f(x)$ are positive and increase without bound as x approaches a from either side (see Figure 1.14(a))
Similarly, we say that **the limit of $f(x)$ as x approaches a is negative infinity**, and write

$$\lim_{x \to a} f(x) = -\infty$$

if the values of $f(x)$ are negative and decrease without bound as x approaches a from either side (see Figure 1.14(b)).

Note Informal definitions of one-sided infinite limits at $x = a$ can also be written in a similar fashion.

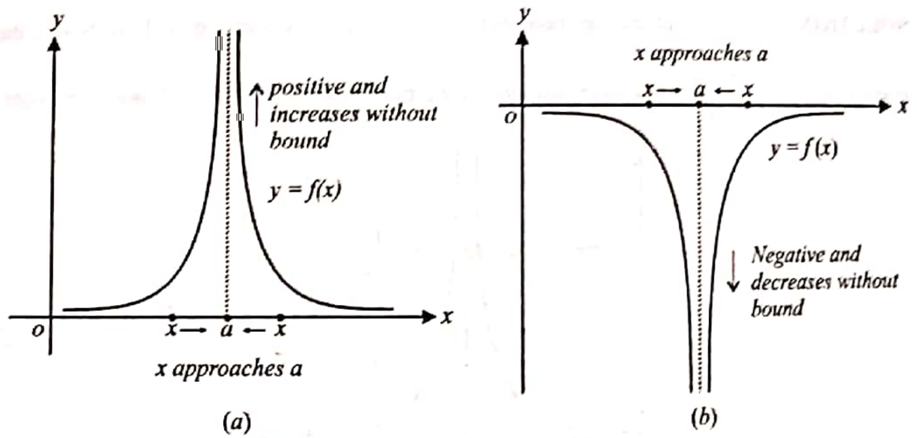

FIGURE 1.14

EXAMPLE 28 Evaluate $\lim_{x \to 2^+} \frac{1}{x-2}$ and $\lim_{x \to 2^-} \frac{1}{x-2}$ using the graph of the function.

SOLUTION The graph of the function $f(x)=\frac{1}{x-2}$ is shown in Figure 1.15. Notice that graph of $f(x)=\frac{1}{x-2}$ is obtained by shifting the graph of $y=\frac{1}{x}$ 2 units to the right. From the graph, we see that the function increases without bound as x approaches 2 from the right and decreases without bound as x approaches 2 from the left. Thus,

$$\lim_{x\to 2^+}\frac{1}{x-2}=\infty \quad\text{and}\quad \lim_{x\to 2^-}\frac{1}{x-2}=-\infty.$$

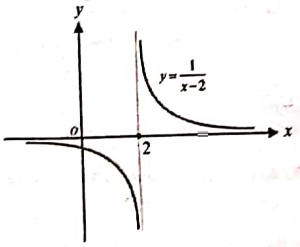

FIGURE 1.15

EXAMPLE 29 Evaluate $\lim_{x\to 1}\frac{1}{(x-1)^2}$ using the graph of the function.

SOLUTION The graph of the function $f(x)=\frac{1}{(x-1)^2}$ is shown in Figure 1.16. Notice that graph of $f(x)=\frac{1}{(x-1)^2}$ can be obtained by shifting the graph of $f(x)=\frac{1}{x^2}$ 1 unit to the right.

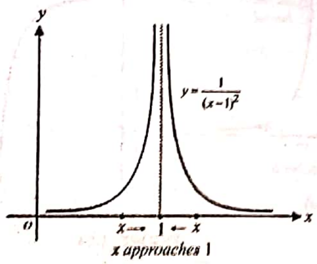

FIGURE 1.16

From the graph, we see that as x approaches 1 from either side, the values of $y=\frac{1}{(x-1)^2}$ are positive and increase without bound. Therefore, $\lim_{x\to 1}\frac{1}{(x-1)^2}=\infty$.

EXAMPLE 30 Evaluate $\lim_{x\to -3}\frac{-1}{(x+3)^2}$ using the graph of the function.

SOLUTION The graph of the function $f(x)=\frac{-1}{(x+3)^2}$ is shown in Figure 1.17. Notice that the graph of $f(x)=\frac{-1}{(x+3)^2}$ can be obtained by shifting the graph of $f(x)=\frac{-1}{x^2}$ 3 units to the left. From the graph, we see that as x approaches -3 from either side, the values of $f(x)=\frac{-1}{(x+3)^2}$ are negative and decrease without bound. Therefore, $\lim_{x\to -3}\frac{-1}{(x+3)^2}=-\infty$.

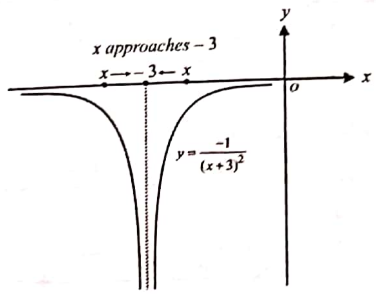

FIGURE 1.17

Precise Definitions of Infinite Limits

DEFINITION Infinite Limits

Let f be a function defined on an open interval containing a, except possibly at a itself. We say that **$f(x)$ approaches infinity as x approaches a**, and write

$$\lim_{x\to a} f(x)=\infty$$

if, for every positive number B, there exists a corresponding $\delta>0$ such that

$$f(x) > B \quad\text{whenever}\quad 0<|x-a|<\delta$$

This means that for any positive number B (no matter how large), the values of $f(x)$ can be made larger than B by making x sufficiently close to a but not equal to a. A geometric illustration is shown in Figure 1.18.

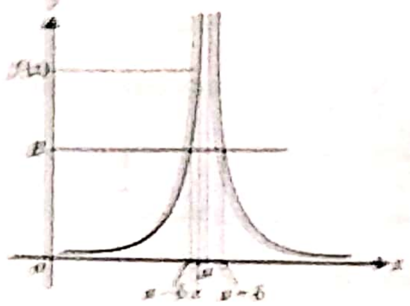

FIGURE 1.18

DEFINITION Infinite Limits

Let f be a function defined on an open interval containing a, except possibly at a itself. We say that the $f(x)$ **approaches negative infinity as x approaches a**, and write

$$\lim_{x \to a} f(x) = -\infty$$

if, for every negative number $-B$, there exists a corresponding $\delta > 0$ such that

$$f(x) < -B \quad \text{whenever} \quad 0 < |x - a| < \delta$$

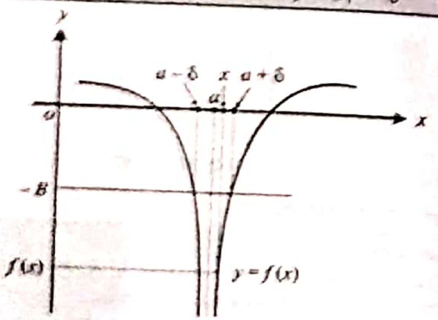

FIGURE 1.19

This means that for any negative number $-B$ (no matter how small), the values of $f(x)$ can be made smaller than $-B$ by making x sufficiently close to a but not equal to a. This is illustrated in Figure 1.19.

Note Precise definitions of one-sided infinite limits at $x = a$ can also be written in a similar fashion.

EXAMPLE 31 Use the definition of infinite limits to prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

SOLUTION Let $B > 0$ be arbitrary. We must find a number $\delta > 0$ such that

$$\frac{1}{x^2} > B \quad \text{whenever} \quad 0 < |x - 0| < \delta$$

Now,

$$\frac{1}{x^2} > B \quad \text{iff} \quad x^2 < \frac{1}{B} \quad \text{iff} \quad |x| < \frac{1}{\sqrt{B}}$$

Thus, choosing $\delta = \frac{1}{\sqrt{B}}$ (or any smaller positive number), we see that

$$\text{if} \quad |x| < \delta = \frac{1}{\sqrt{B}} \quad \text{then} \quad \frac{1}{x^2} > B$$

Thus, by definition, it follows that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

DEFINITION Vertical Asymptote

A line $x = a$ is called a **vertical asymptote** of the graph of a function $y = f(x)$ if one of the following statements is true :

$$\lim_{x \to a^-} f(x) = \infty, \quad \lim_{x \to a^+} f(x) = \infty, \quad \lim_{x \to a^-} f(x) = -\infty, \quad \lim_{x \to a^+} f(x) = -\infty$$

Thus, a line $x = a$ is a vertical asymptote of the graph of f if $f(x)$ approaches infinity (or negative infinity) as x approaches a from the left or the right.

For example, the line $x = 0$ (the y-axis) is a vertical asymptote of the graph of $y = \frac{1}{x}$ because

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

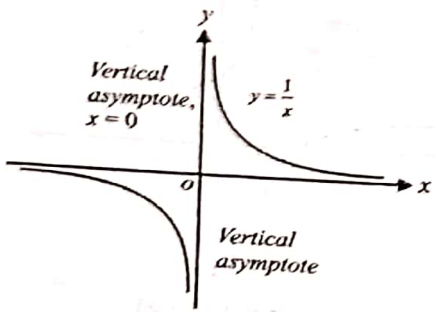

FIGURE 1.20

Notice that the distance between a point on the graph of $y=\frac{1}{x}$ and the y-axis approaches zero as the point moves vertically along the curve and away from the origin (see Figure 1.20).

Finding Vertical Asymptotes of Rational Functions

To find the vertical asymptotes of a rational function, we set the denominator equal to 0 and solve for x. The vertical asymptotes occur at those values of x that produce 0 in the denominator but not in the numerator.

EXAMPLE 32 Find the vertical asymptote of the graph of $f(x)=\frac{x+2}{x+1}$.

SOLUTION To find the vertical asymptote, we need to find the behaviour of $f(x)$ as $x \to -1$, where the denominator is 0.

As $x \to -1^-$, the numerator $x+2$ approaches $(-1)+2=1$ while the denominator $x+1$ is negative and approaches 0. Therefore,

$$\lim_{x \to -1^-} \frac{x+2}{x+1} = -\infty.$$

As $x \to -1^+$, the numerator $x+2$ approaches $(-1)+2=1$ while the denominator $x+1$ is positive and approaches 0. Therefore,

$$\lim_{x \to -1^+} \frac{x+2}{x+1} = \infty$$

The infinite limits $\lim_{x \to -1^-} f(x) = -\infty$ and $\lim_{x \to -1^+} f(x) = \infty$ each imply that the line $x=-1$ is a vertical asymptote of f. The graph of the function $y=\frac{x+2}{x+1}$ is shown in Figure 1.21.

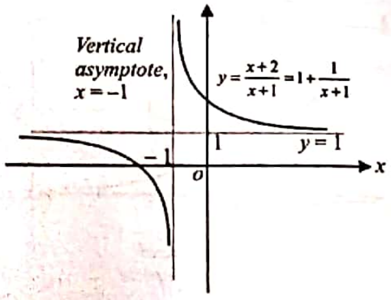

FIGURE 1.21

Notice that, using long division, we can rewrite y as $y=1+\frac{1}{x+1}$. Thus the curve in question is the graph of $y=\frac{1}{x}$ shifted 1 unit up and 1 unit left.

EXAMPLE 33 Find the vertical asymptotes of the graph of $f(x)=\frac{x^2-4x+3}{x^2-1}$.

SOLUTION To find the vertical asymptotes, we need to determine the behaviour of f as $x \to \pm 1$, where the denominator is zero.

***(a)* The behaviour as $x \to 1$.** As $x \to 1$, both the numerator and denominator of f approach 0, and the function is undefined at $x=1$. Since we are not concerned with what happens to the quotient when x equals 1, so we can assume that $x \neq 1$ and write

$$\frac{x^2-4x+3}{x^2-1} = \frac{(x-1)(x-3)}{(x-1)(x+1)} = \frac{x-3}{x+1} \qquad \text{for } x \neq 1$$

$$\therefore \qquad \lim_{x \to 1} \frac{x^2-4x+3}{x^2-1} = \lim_{x \to 1} \frac{x-3}{x+1} = \frac{1-3}{1+1} = -1$$

Thus, $\lim_{x \to 1} f(x) = -1$ (even though f is not defined at $x=1$). The line $x=1$ is *not* a vertical asymptote of f.

***(b)* The behaviour as $x \to -1$.** We just showed that

$$f(x)=\frac{x^2-4x+3}{x^2-1}=\frac{x-3}{x+1} \qquad \text{for } x \neq 1$$

We use this fact again. As $x \to -1^-$, $x-3$ approaches $(-1)-3=-4$ while $x+1$ is negative and approaches 0. Therefore,

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{x-3}{x+1} = \infty$$

Further, as $x \to -1^+$, $x-3$ approaches $(-1)-3=-4$ while $x+1$ is positive and approaches 0. Therefore,

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{x-3}{x+1} = -\infty$$

The infinite limits $\lim_{x \to -1^-} f(x) = \infty$ and $\lim_{x \to -1^+} f(x) = -\infty$ each imply that the line $x=-1$ is a vertical asymptote of f.

EXAMPLE 34 Find the asymptotes of the function $f(x)=\frac{x^2-3}{2x-4}$.

[*Delhi Univ. GE*-1, 2015, 2017]

SOLUTION ***Horizontal Asymptotes :***

To find the horizontal asymptotes, we need to find the behaviour of $f(x)$ as $x \to \pm\infty$. We have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-3}{2x-4} = \lim_{x\to\infty} \frac{\frac{x^2}{x}-\frac{3}{x}}{\frac{2x}{x}-\frac{4}{x}} = \lim_{x\to\infty} \frac{x-\frac{3}{x}}{2-\frac{4}{x}} = \infty$$

A similar analysis shows that

$$\lim_{x\to-\infty} f(x) = -\infty$$

Because these limits are not finite, there are no horizontal asymptote.

Vertical Asymptotes :

To find the vertical asymptotes, we need to find the behaviour of $f(x)$ as $x \to 2$, where the denominator is 0.

As $x \to 2^+$, $x^2 - 3$ approaches $(2)^2 - 3 = 1$ while $2x - 4$ is positive and approaches 0. Therefore,

$$\lim_{x\to 2^+} f(x) = \infty$$

Further, as $x \to 2^-$, $x^2 - 3$ approaches $(2)^2 - 3 = 1$ while $2x - 4$ is negative and approaches 0. Therefore,

$$\lim_{x\to 2^-} f(x) = -\infty$$

The infinite limits $\lim_{x\to 2^+} f(x) = \infty$ and $\lim_{x\to 2^-} f(x) = -\infty$ each imply that the line $x = 2$ is a vertical asymptote of f.

EXERCISE 1.4

Find the vertical and horizontal asymptotes of the following functions.

1. $f(x) = \frac{1}{x-2}$
2. $f(x) = \frac{3x-1}{x+1}$
3. $f(x) = \frac{3-2x}{4-x}$
4. $f(x) = \frac{2x-1}{x^2+4}$
5. $f(x) = \frac{4x+5}{4x^2-9}$
6. $f(x) = \frac{x^2+3}{x^2-4}$
7. $f(x) = \frac{3x+4}{x^2}$
8. $f(x) = \frac{x^3+3x+5}{6x+2}$
9. $f(x) = \frac{4x^2-3}{2x^2-3x+1}$
10. $f(x) = \frac{x^3+2x+1}{x^2-x-12}$
11. $f(x) = \frac{x^2}{\sqrt{x^4+1}}$
12. $f(x) = \frac{x^3-9x}{4x^2+8x}$

ANSWERS

1. VA: $x = 2$; HA: $y = 0$
2. VA: $x = -1$; HA: $y = 3$
3. VA: $x = 4$; HA: $y = 2$
4. VA: none; HA: $y = 0$
5. VA: $x = -3/2$, $x = 3/2$; HA: $y = 0$
6. VA: $x = -2$, $x = 2$; HA: $y = 1$
7. VA: $x = 0$; HA: $y = 0$
8. VA: $x = -1/3$; HA: none
9. VA: $x = 1/2$, $x = 1$; HA: $y = 2$
10. VA: $x = -3$, $x = 4$; HA: none
11. VA: none; HA: $y = 1$
12. VA: $x = -2$; HA: none